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On primitive numerical semigroups of genus 10¹

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Abstract

We investigate whether a primitive numerical semigroup of genus 10 is Weierstrass. There are 89 primitive numerical semigroups of genus 10. We prove that 75 semigroups among them are Weierstrass.

§1. A characterization of a primitive numerical semigroup

Let \mathbb{N}_0 be the additive semigroup of non-negative integers. A subsemigroup H of \mathbb{N}_0 is called a *numerical semigroup* if whose complement $\mathbb{N}_0 \setminus H$ in \mathbb{N}_0 is a finite set. The cardinality of the set $\mathbb{N}_0 \setminus H$ is called the *genus* of H , which is denoted by $g(H)$. Let H be a numerical semigroup. We set $\gamma(H) = \max\{\gamma \in \mathbb{N}_0 \mid \gamma \notin H\}$ and $a(H) = \min\{h > 0 \mid h \in H\}$. A numerical semigroup H is said to be *primitive* if $\gamma(H) < 2a(H)$. A numerical semigroup is called an *n-semigroup* if $a(H) = n$. For $a_1, \dots, a_m \in \mathbb{N}_0$ $\langle a_1, \dots, a_m \rangle$ denotes the semigroup generated by a_1, \dots, a_m .

Example. Let $H = \langle 7, 8, 11, 17, 20 \rangle$. Since $\mathbb{N}_0 \setminus H = \{1 \rightarrow 6, 9, 10, 12, 13\}$, H is a primitive 7-semigroup H of genus 10.

We state some properties of primitive numerical semigroups without proof.

Remark. If H is a primitive n -semigroup of genus g , then we have $\frac{g}{2} + 1 \leq n \leq g + 1$.

Remark. For a subset S of \mathbb{N}_0 the following are equivalent:

- i) S is a primitive n -semigroup.
- ii) $\{1, \dots, n-1\} \subseteq \mathbb{N}_0 \setminus S$ and $\mathbb{N}_0 \setminus S \subseteq \{1, \dots, n-1\} \cup \{n+1, \dots, 2n-1\}$.

We want to investigate the number of the primitive numerical semigroups of genus g .

¹This is an abstract and the details will be published elsewhere.

Proposition. Let $\frac{g}{2} + 1 \leq n \leq g + 1$. Then the number $\mathcal{P}_n(g)$ of the primitive n -semigroups of genus g is $_{n-1}C_{g-(n-1)}$

Proposition. Let $\mathcal{P}(g)$ be the number of the primitive numerical semigroups of genus g . Then we have
$$\sum_{m=\lceil \frac{g+1}{2} \rceil}^g {}_mC_{g-m}.$$

Example. The number of the primitive numerical semigroups of genus 10 is 89, because $\mathcal{P}(10) = \sum_{n=6}^{11} \mathcal{P}_n(10) = \sum_{m=5}^{10} {}_mC_{10-m} = 1 + 15 + 35 + 28 + 9 + 1 = 89$.

§2. On the weight of a primitive numerical semigroup

We are interested in primitive numerical semigroups of genus 10. We want to explain its reason. In this paper a *curve* C means a complete non-singular irreducible curve over an algebraically closed field k of characteristic 0. For a point P of C we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\}$$

where $k(C)$ denotes the field of rational functions on C and $(f)_\infty$ is the polar divisor of the function f .

Definition. A numerical semigroup H is said to be *Weierstrass* if there is a pointed curve (C, P) such that $H = H(P)$.

Fact. Every primitive numerical semigroup of genus $g \leq 9$ is Weierstrass ($g = 4$: [9], $5 \leq g \leq 8$: [5], $g = 9$: [6]).

We know that a *generic* primitive numerical semigroup is Weierstrass. We want to explain the meaning of *generic*. Let C be a curve of genus g . For every point P of C except the Weierstrass points we have $H(P) = \langle g + 1, \dots, 2g + 1 \rangle$, i.e., $\mathbb{N}_0 \setminus H(P) = \{1, \dots, g\}$. We note that the number of the Weierstrass points is less than or equal to $(g - 1)g(g + 1)$. We introduce the notion of the *weight* $w(H)$ of a numerical semigroup H of genus g to indicate the difference between the semigroup $\langle g + 1, \dots, 2g + 1 \rangle$ and H as follows: Let $\mathbb{N}_0 \setminus H = \{\gamma_1 < \dots < \gamma_g\}$. Then we set $w(H) = \sum_{i=1}^g (\gamma_i - i)$. So, the smaller is the weight $w(H)$, the more generic is H .

Fact. If H is a primitive numerical semigroup with $w(H) \leq g(H) - 1$, then it is Weierstrass ($w(H) \leq g(H) - 2$: [2], $w(H) = g(H) - 1$: [4]).

Thus, it is sufficient to study primitive numerical semigroups H with $w(H) \geq g(H)$ when we investigate whether a primitive numerical semigroup is Weierstrass or not. Here we give the table of the number of primitive numerical semigroups H of genus 10 classified by the minimum positive integer n in H and the weight of H .

n	$\mathcal{P}_n(10)$	$w(H) \leq 9$	$w(H) \geq 10$
6	1	1	0
7	15	11	4
8	35	20	15
9	28	19	9
10	9	9	0
11	1	1	0
total	89	61	28

Moreover, we give all primitive numerical semigroups of genus 10 with $w(H) \geq 10$.

H	$\mathbb{N}_0 \setminus H$	$w(H)$
$\langle 7, 8, 9, 19, 20 \rangle$	$\{1 \rightarrow 6, 10, 11, 12, 13\}$	12
$\langle 7, 8, 10, 19 \rangle$	$\{1 \rightarrow 6, 9, 11, 12, 13\}$	11
$\langle 7, 8, 11, 17, 20 \rangle$	$\{1 \rightarrow 6, 9, 10, 12, 13\}$	10
$\langle 7, 9, 10, 15 \rangle$	$\{1 \rightarrow 6, 8, 11, 12, 13\}$	10
$\langle 8, 9, 10, 11, 12 \rangle$	$\{1 \rightarrow 7, 13, 14, 15\}$	15
$\langle 8, 9, 10, 11, 13 \rangle$	$\{1 \rightarrow 7, 12, 14, 15\}$	14
$\langle 8, 9, 10, 11, 14 \rangle$	$\{1 \rightarrow 7, 12, 13, 15\}$	13
$\langle 8, 9, 10, 11, 15 \rangle$	$\{1 \rightarrow 7, 12, 13, 14\}$	12
$\langle 8, 9, 10, 12, 13 \rangle$	$\{1 \rightarrow 7, 11, 14, 15\}$	13
$\langle 8, 9, 10, 12, 14 \rangle$	$\{1 \rightarrow 7, 11, 13, 15\}$	12
$\langle 8, 9, 10, 12, 15 \rangle$	$\{1 \rightarrow 7, 11, 13, 14\}$	11
$\langle 8, 9, 10, 13, 14 \rangle$	$\{1 \rightarrow 7, 11, 12, 15\}$	11
$\langle 8, 9, 10, 13, 15 \rangle$	$\{1 \rightarrow 7, 11, 12, 14\}$	10
$\langle 8, 9, 11, 12, 13 \rangle$	$\{1 \rightarrow 7, 10, 14, 15\}$	12
$\langle 8, 9, 11, 12, 14 \rangle$	$\{1 \rightarrow 7, 10, 13, 15\}$	11
$\langle 8, 9, 11, 12, 15 \rangle$	$\{1 \rightarrow 7, 10, 13, 14\}$	10
$\langle 8, 9, 11, 13, 14 \rangle$	$\{1 \rightarrow 7, 10, 12, 15\}$	10

H	$\mathbb{N}_0 \setminus H$	$w(H)$
$\langle 8, 10, 11, 12, 13, 17 \rangle$	$\{1 \longrightarrow 7, 9, 14, 15\}$	11
$\langle 8, 10, 11, 12, 14, 17 \rangle$	$\{1 \longrightarrow 7, 9, 13, 15\}$	10
$\langle 9, 10, 11, 12, 13, 14, 15 \rangle$	$\{1 \longrightarrow 8, 16, 17\}$	14
$\langle 9, 10, 11, 12, 13, 14, 16 \rangle$	$\{1 \longrightarrow 8, 15, 17\}$	13
$\langle 9, 10, 11, 12, 13, 14, 17 \rangle$	$\{1 \longrightarrow 8, 15, 16\}$	12
$\langle 9, 10, 11, 12, 13, 15, 16 \rangle$	$\{1 \longrightarrow 8, 14, 17\}$	12
$\langle 9, 10, 11, 12, 13, 15, 17 \rangle$	$\{1 \longrightarrow 8, 14, 16\}$	11
$\langle 9, 10, 11, 12, 13, 16, 17 \rangle$	$\{1 \longrightarrow 8, 14, 15\}$	10
$\langle 9, 10, 11, 12, 14, 15, 16 \rangle$	$\{1 \longrightarrow 8, 13, 17\}$	11
$\langle 9, 10, 11, 12, 14, 15, 17 \rangle$	$\{1 \longrightarrow 8, 13, 16\}$	10
$\langle 9, 10, 11, 13, 14, 15, 16 \rangle$	$\{1 \longrightarrow 8, 12, 17\}$	10

§3. On the Schubert index of a primitive numerical semigroup

We introduce a new notion for describing primitive numerical semigroups. Let $0 \leq \alpha_1 \leq \dots \leq \alpha_g \leq g-1$ be integers. $\alpha = (\alpha_1, \dots, \alpha_g)$ is called a *Schubert index of genus g* . The number $\sum_{i=1}^g \alpha_i$ is called the *weight* of α , which is denoted by $w(\alpha)$. We set $H(\alpha) = \mathbb{N}_0 \setminus G(\alpha)$ where $G(\alpha) = \{\alpha_1 + 1, \alpha_2 + 2, \dots, \alpha_g + g\}$. The Schubert index α is *primitive* if $H(\alpha)$ is a primitive numerical semigroup. Let H be a numerical semigroup of genus g . Set $\mathbb{N}_0 \setminus H = \{\gamma_1, \dots, \gamma_g\}$ and $\alpha(H) = (\gamma_1 - 1, \dots, \gamma_g - g)$. Then $\alpha(H)$ is a Schubert index of genus g . Moreover, $w(H) = w(\alpha(H))$.

Remark. If H is a primitive numerical semigroup, then $\alpha(H)$ is a primitive Schubert index.

Example. For an ordinary point P , i.e., a non-Weierstrass point, of a curve C of genus g we have $\alpha(H(P)) = (0, \dots, 0) = (0^g)$, because $H(P) = \langle g+1, g+2, \dots, 2g+1 \rangle$ implies that $\mathbb{N}_0 \setminus H(P) = \{1, 2, \dots, g\}$.

We want to define a partial order on the set of primitive Schubert indices.

Definition. Let $\alpha = (\alpha_1, \dots, \alpha_{g-1})$ and $\beta = (\beta_1, \dots, \beta_g)$ be primitive Schubert indices of genus $g-1$ and g respectively. We define $\alpha \implies \beta$ if one of the following holds.

- i) $\alpha_i = \beta_{i+1}$ for any $1 \leq i \leq g-1$,
- ii) $1 \leq \exists j \leq g-1$ such that $\alpha_j = \beta_{j+1} - 1$ and $\alpha_i = \beta_{i+1}$ for $\forall i \neq j$.

Example. i) $(0^7, 6, 6) \Rightarrow (0^8, 6, 6)$. ii) $(0^6, 2, 4, 4) \Rightarrow (0^7, 2, 4, 5)$.

iii) $(0^6, 3, 3, 3) \Rightarrow (0^6, 1, 3, 3, 3)$.

Definition. Let α and β be primitive Schubert indices. We define $\alpha \leq \beta$ if $\alpha = \beta$ or \exists a sequence $\alpha \Rightarrow \alpha^{(n)} \Rightarrow \dots \Rightarrow \alpha^{(1)} \Rightarrow \beta$.

Example. i) $(0^5, 2, 3, 3) \leq (0^6, 1, 3, 3, 3)$, because

$$(0^5, 2, 3, 3) \Rightarrow (0^6, 3, 3, 3) \Rightarrow (0^6, 1, 3, 3, 3).$$

ii) $(0^4, 3, 3) \leq (0^8, 4, 6)$, because

$$(0^4, 3, 3) \Rightarrow (0^5, 3, 4) \Rightarrow (0^6, 3, 5) \Rightarrow (0^7, 4, 5) \Rightarrow (0^8, 4, 6).$$

Definition. A primitive Schubert index β is *minimal* if $\nexists \alpha$ such that $\alpha \leq \beta$ and $\beta \neq \alpha$.

Example. $(0^4, 3, 3)$, $(0^5, 2, 2, 2, 2)$ and $(0^6, 2, 2, 3, 3)$ are minimal.

Remark. Let α be a primitive Schubert index of genus g .

i) If $w(\alpha) \leq g - 2$, then $(0) \leq \alpha$.

ii) If $w(\alpha) = g - 1$, then \exists an odd h such that $(0^{\frac{h+1}{2}}, 2^{\frac{h-1}{2}}) \leq \alpha$.

We give the Schubert indices and the properties of all primitive numerical semigroups H of genus 10 with $w(H) \geq 10$.

H	$\alpha(H)$	Property	$w(\alpha(H))$
$\langle 7, 8, 9, 19, 20 \rangle$	$(0^6, 3^4)$	<i>minimal</i>	12
$\langle 7, 8, 10, 19 \rangle$	$(0^6, 2, 3^3)$	<i>minimal</i>	11
$\langle 7, 8, 11, 17, 20 \rangle$	$(0^6, 2^2, 3^2)$	<i>minimal</i>	10
$\langle 7, 9, 10, 15 \rangle$	$(0^6, 1, 3^3)$	$\geq (0^5, 2, 3^2)$	10
$\langle 8, 9, 10, 11, 12 \rangle$	$(0^7, 5^3)$	<i>minimal</i>	15
$\langle 8, 9, 10, 11, 13 \rangle$	$(0^7, 4, 5^2)$	<i>minimal</i>	14
$\langle 8, 9, 10, 11, 14 \rangle$	$(0^7, 4^2, 5)$	$\geq (0^6, 4^3)$	13
$\langle 8, 9, 10, 11, 15 \rangle$	$(0^7, 4^3)$	$\geq (0^6, 3, 4^2)$	12
$\langle 8, 9, 10, 12, 13 \rangle$	$(0^7, 3, 5^2)$	<i>minimal</i>	13
$\langle 8, 9, 10, 12, 14 \rangle$	$(0^7, 3, 4, 5)$	$\geq (0^6, 3, 4^2)$	12
$\langle 8, 9, 10, 12, 15 \rangle$	$(0^7, 3, 4^2)$	$\geq (0^5, 3^3)$	11
$\langle 8, 9, 10, 13, 14 \rangle$	$(0^7, 3^2, 5)$	$\geq (0^5, 3^3)$	11
$\langle 8, 9, 10, 13, 15 \rangle$	$(0^7, 3^2, 4)$	$\geq (0^5, 2, 3^2)$	10

H	$\alpha(H)$	$Property$	$w(\alpha(H))$
$\langle 8, 9, 11, 12, 13 \rangle$	$(0^7, 2, 5^2)$	<i>minimal</i>	12
$\langle 8, 9, 11, 12, 14 \rangle$	$(0^7, 2, 4, 5)$	$\geq (0^6, 2, 4^2)$	11
$\langle 8, 9, 11, 12, 15 \rangle$	$(0^7, 2, 4, 4)$	$\geq (0^4, 3^2)$	10
$\langle 8, 9, 11, 13, 14 \rangle$	$(0^7, 2, 3, 5)$	$\geq (0^5, 2, 3^2)$	10
$\langle 8, 10, 11, 12, 13, 17 \rangle$	$(0^7, 1, 5^2)$	$\geq (0^5, 4^2)$	11
$\langle 8, 10, 11, 12, 14, 17 \rangle$	$(0^7, 1, 4, 5)$	$\geq (0^4, 3^2)$	10
$\langle 9, 10, 11, 12, 13, 14, 15 \rangle$	$(0^8, 7^2)$	<i>minimal</i>	14
$\langle 9, 10, 11, 12, 13, 14, 16 \rangle$	$(0^8, 6, 7)$	$\geq (0^7, 6^2)$	13
$\langle 9, 10, 11, 12, 13, 14, 17 \rangle$	$(0^8, 6^2)$	$\geq (0^6, 5^2)$	12
$\langle 9, 10, 11, 12, 13, 15, 16 \rangle$	$(0^8, 5, 7)$	$\geq (0^6, 5^2)$	12
$\langle 9, 10, 11, 12, 13, 15, 17 \rangle$	$(0^8, 5, 6)$	$\geq (0^5, 4^2)$	11
$\langle 9, 10, 11, 12, 13, 16, 17 \rangle$	$(0^8, 5^2)$	$\geq (0^4, 3^2)$	10
$\langle 9, 10, 11, 12, 14, 15, 16 \rangle$	$(0^8, 4, 7)$	$\geq (0^5, 4^2)$	11
$\langle 9, 10, 11, 12, 14, 15, 17 \rangle$	$(0^8, 4, 6)$	$\geq (0^4, 3^2)$	10
$\langle 9, 10, 11, 13, 14, 15, 16 \rangle$	$(0^8, 3, 7)$	$\geq (0^4, 3^2)$	10

§4. On the moduli space \mathcal{M}_H

Definition. Let $\mathcal{M}_{g,1}$ be the moduli space of pointed curves of genus g . Let α be a primitive Schubert index of genus g . We set

$$\mathcal{C}_\alpha = \{(C, P) \in \mathcal{M}_{g,1} \mid H(P) = H(\alpha)\}.$$

We say that α is *dimensionally proper* if there is an open subset U of $\mathcal{M}_{g,1}$ such that $\mathcal{C}_\alpha \cap U$ is a non-empty set with codimension $w(\alpha)$ in U .

Fact. Let $\alpha \leq \beta$ be primitive Schubert indices. If α is dimensionally proper, so is β ([2]).

Remark. The primitive Schubert index (0^g) of genus g is dimensionally proper. Hence, if α is a primitive Schubert index of genus g with $w(\alpha) \leq g-2$, then it is dimensionally proper, which implies that every primitive numerical semigroup of genus g with $w(H) \leq g-2$ is Weierstrass.

Fact. i) For any odd h the index $(0^{\frac{h+1}{2}}, 2^{\frac{h-1}{2}})$ is dimensionally proper. Hence, every primitive numerical semigroup H of genus g with $w(H) = g-1$ is Weierstrass ([4]).

ii) $(0^4, 3^2)$ is dimensionally proper ([10],[5]).

- iii) $(0^5, 4^2)$ is dimensionally proper ([10]).
 iv) $(0^6, 5^2)$ is dimensionally proper ([6]).

Fact. If H is a numerical semigroup of genus g with $\alpha(H) = (0^n, l^{g-n})$, then it is Weierstrass ([3]).

Using the above facts we get the following table which shows that certain semigroups are Weierstrass. In the table below Δ means that the semigroup will be proved to be Weierstrass in the forthcoming paper.

H	$\alpha(H)$	Property	$w(\alpha(H))$	Weierstrass
$\langle 7, 8, 9, 19, 20 \rangle$	$(0^6, 3^4)$	<i>minimal</i>	12	\bigcirc
$\langle 7, 8, 10, 19 \rangle$	$(0^6, 2, 3^3)$	<i>minimal</i>	11	?
$\langle 7, 8, 11, 17, 20 \rangle$	$(0^6, 2^2, 3^2)$	<i>minimal</i>	10	?
$\langle 7, 9, 10, 15 \rangle$	$(0^6, 1, 3^3)$	$\geq (0^5, 2, 3^2)$	10	?
$\langle 8, 9, 10, 11, 12 \rangle$	$(0^7, 5^3)$	<i>minimal</i>	15	\bigcirc
$\langle 8, 9, 10, 11, 13 \rangle$	$(0^7, 4, 5^2)$	<i>minimal</i>	14	Δ
$\langle 8, 9, 10, 11, 14 \rangle$	$(0^7, 4^2, 5)$	$\geq (0^6, 4^3)$	13	Δ
$\langle 8, 9, 10, 11, 15 \rangle$	$(0^7, 4^3)$	$\geq (0^6, 3, 4^2)$	12	\bigcirc
$\langle 8, 9, 10, 12, 13 \rangle$	$(0^7, 3, 5^2)$	<i>minimal</i>	13	?
$\langle 8, 9, 10, 12, 14 \rangle$	$(0^7, 3, 4, 5)$	$\geq (0^6, 3, 4^2)$	12	?
$\langle 8, 9, 10, 12, 15 \rangle$	$(0^7, 3, 4^2)$	$\geq (0^5, 3^3)$	11	Δ
$\langle 8, 9, 10, 13, 14 \rangle$	$(0^7, 3^2, 5)$	$\geq (0^5, 3^3)$	11	?
$\langle 8, 9, 10, 13, 15 \rangle$	$(0^7, 3^2, 4)$	$\geq (0^5, 2, 3^2)$	10	?
$\langle 8, 9, 11, 12, 13 \rangle$	$(0^7, 2, 5^2)$	<i>minimal</i>	12	?
$\langle 8, 9, 11, 12, 14 \rangle$	$(0^7, 2, 4, 5)$	$\geq (0^6, 2, 4^2)$	11	?
$\langle 8, 9, 11, 12, 15 \rangle$	$(0^7, 2, 4, 4)$	$\geq (0^4, 3^2)$	10	\bigcirc
$\langle 8, 9, 11, 13, 14 \rangle$	$(0^7, 2, 3, 5)$	$\geq (0^5, 2, 3^2)$	10	Δ
$\langle 8, 10, 11, 12, 13, 17 \rangle$	$(0^7, 1, 5^2)$	$\geq (0^5, 4^2)$	11	\bigcirc
$\langle 8, 10, 11, 12, 14, 17 \rangle$	$(0^7, 1, 4, 5)$	$\geq (0^4, 3^2)$	10	\bigcirc
$\langle 9, 10, 11, 12, 13, 14, 15 \rangle$	$(0^8, 7^2)$	<i>minimal</i>	14	\bigcirc
$\langle 9, 10, 11, 12, 13, 14, 16 \rangle$	$(0^8, 6, 7)$	$\geq (0^7, 6^2)$	13	Δ
$\langle 9, 10, 11, 12, 13, 14, 17 \rangle$	$(0^8, 6^2)$	$\geq (0^6, 5^2)$	12	\bigcirc
$\langle 9, 10, 11, 12, 13, 15, 16 \rangle$	$(0^8, 5, 7)$	$\geq (0^6, 5^2)$	12	\bigcirc
$\langle 9, 10, 11, 12, 13, 15, 17 \rangle$	$(0^8, 5, 6)$	$\geq (0^5, 4^2)$	11	\bigcirc
$\langle 9, 10, 11, 12, 13, 16, 17 \rangle$	$(0^8, 5^2)$	$\geq (0^4, 3^2)$	10	\bigcirc
$\langle 9, 10, 11, 12, 14, 15, 16 \rangle$	$(0^8, 4, 7)$	$\geq (0^5, 4^2)$	11	\bigcirc

H	$\alpha(H)$	Property	$w(\alpha(H))$	Weierstrass
$\langle 9, 10, 11, 12, 14, 15, 17 \rangle$	$(0^8, 4, 6)$	$\geq (0^4, 3^2)$	10	○
$\langle 9, 10, 11, 13, 14, 15, 16 \rangle$	$(0^8, 3, 7)$	$\geq (0^4, 3^2)$	10	○

Let H_1 and H_2 be primitive numerical semigroups. We define $H_1 \leq H_2$ by $\alpha(H_1) \leq \alpha(H_2)$.

Problem 1. Let H_0 be a Weierstrass primitive semigroup. If H is a primitive numerical semigroup with $H_0 \leq H$, then is it Weierstrass ? i.e., $\forall H_0 = \text{“Wei”} \leq H = \text{“Wei”}$?

Definition. Let H be a numerical semigroup of genus g . For any $m \geq 2$ we set

$$L_m(H) = \{\gamma_1 + \cdots + \gamma_m \mid \gamma_i \in \mathbb{N}_0 \setminus H, \forall i\}.$$

We say that H is *quasi-Weierstrass* if $\#L_m(H) \leq (2m-1)(g-1)$ for $\forall m \geq 2$.

Fact. If a numerical semigroup H is Weierstrass, then it is quasi-Weierstrass ([1]).

Example A. There exists a primitive numerical semigroup H which is not quasi-Weierstrass such that $H_0 \leq H$ for some quasi-Weierstrass primitive numerical semigroup H_0 , i.e., $\exists H_0 = \text{“quasi-Wei”} \leq \exists H = \text{“not quasi-Wei”}$.

Example B. There are non-Weierstrass semigroups H which are quasi-Weierstrass ([11]). Up to now all such semigroups H are not primitive.

Problem 2. Every quasi-Weierstrass primitive semigroup is Weierstrass ?

Either Problem 1 or Problem 2 is at false, because of Example A. Finally we give the table which shows the situation on our problem ([5], [8], [6], [7]).

Genus	Weierstrass	Unknown
$g \leq 8$	All	None
$g = 9 \wedge \text{primitive}$	All(55)	None
$g = 9 \wedge \text{non-primitive}$	61	2
$g = 10 \wedge \text{primitive}$	75	14

References

- [1] R.O. Buchweitz, *On Zariski's criterion for equisingularity and non-smoothable monomial curves*, preprint 113, University of Hannover, 1980.
- [2] D. Eisenbud and J. Harris, *Existence, decomposition, and limits of certain Weierstrass points*, Invent. Math. **87** (1987) 495-515.
- [3] S.J. Kim, *On the existence of Weierstrass gap sequences on trigonal curves*, J. Pure Appl. Algebra **63** (1990) 171-180.
- [4] J. Komeda, *On primitive Schubert indices of genus g and weight $g - 1$* , J. Math. Soc. Japan **43** (1991) 437-445.
- [5] J. Komeda, *On the existence of Weierstrass gap sequences on curves of genus ≤ 8* , J. Pure Appl. Algebra **97** (1994) 51-71.
- [6] J. Komeda, *Existence of the primitive Weierstrass gap sequences on curves of genus 9*, Bol. Soc. Bras. Mat. **30** (1999) 125-137.
- [7] J. Komeda, *On numerical semigroups of genus 9*, 数理解析研究所講究録 **1503** (2006) 70-75.
- [8] J. Komeda, and A. Ohbuchi, *Existence of the non-primitive Weierstrass gap sequences on curves of genus 8*, Submitted.
- [9] R. Lax, *Gap sequences and moduli in genus 4*, Math. Z. **175** (1980) 67-75.
- [10] K.O. Stöhr and P. Viana, *Weierstrass gap sequences and moduli varieties of trigonal curves*, J. Pure Appl. Algebra **81** (1992) 63-82.
- [11] F. Torres, *Weierstrass points and double coverings of curves with application: Symmetric numerical semigroups which cannot be realized as Weierstrass semigroups*, Manuscripta Math. **83** (1994) 39-58.